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Geometrization of Linear Perturbation Theory for Diffeomorphism-Invariant Covariant Field Equations. II. Basic Gauge-Invariant Variables with Applications to de Sitter Space-Time

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In a companion paper, a systematic treatment of linearized perturbations and a new geometric definition of gauge-invariant variables, based on the theory of vector bundles and applicable to the case of an arbitrary system of covariant field equations, were carefully presented. One of the purposes of the present paper is to specify a necessary and sufficient condition that a given, finite set of gauge-invariant variables, denoted collectively by ω and referred to as the complete set of basic variables, can be used to extract the equivalence classes of perturbations from ω in a unique way. The above set is complete because it has the following property: a knowledge of ω is all one needs in the sense that if x represents an arbitrary point of the "space-time" manifold X and G denotes any gauge-invariant tensor field on X, then the value of G at $x \in X$ is uniquely specified by giving the germs of basic gauge-invariant variables at $x \in X$. Arguments are proposed that ω also has a stronger property which is more immediately useful: any G is obtainable directly from the basic variables through purely algebraic and differential operations. These results are of practical interest, and one concrete setting where one is led to the explicit definition of ω occurs when considering the infinitesimal perturbation of the metric tensor itself (pure gravity) defined on a fixed background de Sitter space-time and obeying the linearized empty-space Einstein equations with nonnegative cosmological constant Λ; the case $\Lambda = 0$ corresponds to linear perturbation theory in Minkowski space-time.

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1. INTRODUCTION

Based on a geometrical foundation, in Banach and Piekarski (1997), we formulated linear perturbation theory for an arbitrary system of diffeomorphism-invariant, covariant field equations. In particular, we discussed the gauge problem, which has been known for a long time and whose consideration has led to so much of the current work in cosmology (Bardeen, 1980; Ellis and Bruni, 1989; Mukhanov *et al.*, 1992), and examined, from essentially new points of view, the notion of a gauge-invariant variable of order r. Among many other things, it was possible to present the construction of various gauge-invariant variables without ever specifying the detailed form of covariant field equations and without ever analyzing the symmetry properties, if any, of the background "space-time" geometry chosen. Consequently, this construction can be used as an organizing principle for the development of any specific perturbation theory.

In this paper we continue the systematic investigation into the geometrical structure and gauge-invariant foundations of linear perturbation theory for an arbitrary system of covariant field equations. Therefore, it should not be surprising that, to avoid the risk of appearing to be repetitive and even trite, *without further comment* we shall use here those symbols and notions which either appear for the first time in Banach and Piekarski (1997) or are reasonably standard, and the analysis proceeds in a way similar to that already made familiar. Recalling the viewpoint presented in Banach and Piekarski (1997, Sections 3.1 and 4.1), one can define the gauge-invariant perturbation $[s'] \in \Gamma/\Gamma_L$ associated with $s' \in \Gamma$ as the equivalence class of tangents $\bar{s}' \in \Gamma$, *not necessarily satisfying the linearized field equations,* which are equivalent to $s' \in \Gamma$. This definition does not tell us directly how to use $[s']$ in practical calculations, or whether such calculations are possible at all. Therefore, it is natural to ask: Given the notion of a gauge-invariant variable of order r (Banach and Piekarski, 1997, Sections 4.2 and 4.3), could a *finite* set of *basic* gauge-invariant variables, denoted collectively by ω and referred to as the *complete* set, be constructed such that the equivalence classes of perturbations are determined from this set and vice versa? Or, more precisely, will it be possible to extract $[s']$ from ω in a unique way? Another interesting question is alluded to in Banach and Piekarski (1997, Section 6). Could one show, in some suitable way, that any gauge-invariant quantity is obtainable *directly* from the aforementioned basic variables through *purely* local (i.e., algebraic and differential) operations?

These and similar questions will be the subject of this paper. In particular, we shall specify a necessary and sufficient condition that a given, finite set of gauge-invariant variables, denoted as before by ω, forms a (minimal) complete set. For brevity, it will be convenient to name the elements of this set the basic gauge-invariant variables. When fully exploiting the theory based on ω as a fundamental concept for the description of infinitesimal perturbations in covariant field theories, one is necessarily led to the following crucial statement about ω: a knowledge of ω is all one needs in the sense that if x represents an arbitrary point of the manifold X of independent variables[3] and G denotes any gauge-invariant tensor field defined on X (Banach and Piekarski, 1997, Section 4.3), then the value of G at $x \in X$ is uniquely specified by giving the germs (Choquet-Bruhat *et al.*, 1989) of basic gauge-invariant variables at $x \in X$. For reasons to be explained in Section 3.2, we conjecture that this statement implies a stronger one which is more immediately useful: any G is obtainable directly from the basic variables through purely algebraic and differential operations. In the generality maintained here, the set ω has two further useful properties. First, the equivalence classes of perturbations are determined from ω and conversely, so the description of $[s']$ in terms of ω is unique. Second, a full set of linear "propagation" equations can be derived that involves only ω. These equations are physically more transparent than the usual ones, because spurious "gauge mode" solutions are automatically excluded. Of course, such a formulation of linear perturbation theory for covariant field equations will be effective only if we verify that the set ω *indeed* consists of finitely many elements. Fortunately, it is possible to confirm in detail the truth of this property of ω in a variety of cases, and when considering Einstein's gravity theory (Misner *et al.*, 1973) or other metric theories of gravity (Brans and Dicke, 1961; Bergmann, 1968; Wagoner, 1970; Hellings and Nordtvedt, 1973), explicit and comparatively simple examples of ω can be given for homogeneous and isotropic cosmological "background" models (Banach and Piekarski, 1996a–c). Nontrivial extensions to homogeneous but anisotropic cosmological background models (Ryan and Shepley, 1975) are also possible and will be carefully presented elsewhere.

The simple and still interesting example of ω, which is the one analyzed here, is that arising from consideration of the infinitesimal perturbation of the metric tensor itself (pure gravity) defined on a fixed background *de Sitter space-time* (Hawking and Ellis, 1973, p. 124) and obeying the linearized empty-space Einstein equations with nonnegative cosmological constant Λ; the case $\Lambda = 0$ corresponds to linear perturbation theory in Minkowski space-time. The detailed discussion of this example satisfies one of our basic purposes: to present an explanation of how some concrete (and thus specific)

[3] Often the manifold X can be identified with the space-time manifold, even though this interpretation of X is not forced on us.

gauge-invariant variables relate to the general and universal structures of linear perturbation theory for covariant field equations.

Here we proceed as follows. To prepare for the discussion and to further analyze the properties of "infinitesimal diffeomorphism-invariant objects," Section 2 introduces a number of elementary processes for creating new gauge-invariant variables from old, the others being only combinations of them. Section 3 considers, from many points of view, the notion of a complete set of basic gauge-invariant variables. Section 4 shows how the equations governing linearized perturbations look when reexpressed in a manifestly gauge-invariant form. Section 5 is devoted to the explicit construction of ω for the aforementioned case of a fixed background de Sitter space-time. Section 6 is for discussion and conclusion. Finally, the auxiliary technical material is included as an Appendix.

2. OPERATIONS ON GAUGE-INVARIANT VARIABLES

As noted already, this section describes a number of elementary processes for creating new gauge-invariant variables from old, thereby enabling us to discuss the structure of linear perturbation theory for covariant field equations from still another viewpoint. Thus, let $x \mapsto G(x, f_{(r)}, [s'])$ be a gauge-invariant variable of order r, and suppose that this gauge-invariant variable is a sufficiently smooth tensor field on X. The term "sufficiently smooth" means that $x \mapsto G(x, f_{(r)}, [s'])$ is a C^k cross section of T (k sufficiently large), with T denoting the tensor bundle (Banach and Piekarski, 1997, Section 4.3). Clearly, given $G(\cdot, f_{(r)}, [s'])$, we can construct new gauge-invariant variables by a variety of operations. One such operation is accomplished by taking the covariant derivative of $x \mapsto G(x, f_{(r)}, [s'])$ relative to an arbitrary, fixed, linear connection on X (Banach and Piekarski, 1997, Section 2.2). As it will be demonstrated below, the object so obtained, denoted for brevity by $x \mapsto (\nabla G(\cdot, f_{(r)}, [s']))_x$, may be considered as the gauge-invariant variable of order $r + 1$.

To present the essential points in the simplest possible way, the workings of the proof will be illustrated by assuming that $G(\cdot, f_{(r)}, [s'])$ is a scalar gauge-invariant variable ($T = \mathcal{R}$), and the treatment of the general case ($T \neq \mathcal{R}$), while certainly possible, is formally too elaborate for the present work. First of all, from the definition of $\Gamma^*_{(r),F}$ in Banach and Piekarski (1997, Section 4.2) it follows that $f_{(r)} \in \Gamma^*_{(r),F}$ can be decomposed as

$$f_{(r)} = \bigoplus_{p=0}^{r} \left(\bigoplus_{A=1}^{n} f_{A,p} \right) \tag{2.1}$$

where each $f_{A,p}$ is a cross section of the tensor bundle S_p^{A*}. Here it should perhaps be noticed that the tensor bundle S_p^{A*} was originally introduced to define S_p^* and $S_{(r)}^*$ by equations (2.15b) and (2.16b) appearing in Banach and Piekarski (1997), and that the possibility to describe $f_{(r)}$ in terms of $f_{A,p}$ is a direct consequence of these equations. In exactly the same way it will be immediate to find the explicit form of $D^r s'$. As a matter of fact, this form emerges when we put the decomposition

$$s' = \bigoplus_{A=1}^{n} (s^A)' \tag{2.2}$$

of $s' \in \Gamma$ into the decomposition

$$D^r s' = \bigoplus_{p=0}^{r} \nabla^p s' \tag{2.3}$$

of $D^r s'$, so obtaining

$$D^r s' = \bigoplus_{p=0}^{r} \left(\bigoplus_{A=1}^{n} \nabla^p (s^A)' \right) \tag{2.4}$$

Obviously, in the above formula for $D^r s'$ each $\nabla^p (s^A)'$ is a cross section of S_p^A, the tensor bundle dual to $S_p^A{}^*$. Now, using equations (2.1) and (2.4), we conclude from the definition of a scalar gauge-invariant variable of order r [Banach and Piekarski, 1997, equation (4.5)] that $G(\cdot, f_{(r)}, [s'])$ is explicitly given by

$$G(\cdot, f_{(r)}, [s']) = \sum_{p=0}^{r} \sum_{A=1}^{n} f_{A,p} \odot \nabla^p (s^A)' \tag{2.5}$$

in a (hopefully) obvious notation.

For the purposes of this discussion, it will be convenient to regard the covariant derivative ∇ with respect to an arbitrary, fixed, linear connection on X as a "symbolic covariant vector field" $\nabla = \Sigma_{a=1}^{N} e^a \nabla_a$, where[4] $\{e^a\}$ is a frame of the tensor bundle T of type (0, 1) over X (Dieudonné, 1972, p. 119). Consequently, the action of derivative operator ∇ on an arbitrary tensor field B can be characterized by

$$\nabla B = \sum_{a=1}^{N} e^a \otimes \nabla_a B \tag{2.6}$$

Next, let $\{e_a\}$ be a frame of T^* dual the frame $\{e^a\}$ of T. By use of the "unit tensor field"

$$J := \sum_{a=1}^{N} e^a \otimes e_a \tag{2.7}$$

[4] We recall that N is the dimension of X.

on X, we then obtain from equation (2.5) the following expression for the covariant derivative of $G(\cdot, f_{(r)}, [s'])$:

$$\nabla G(\cdot, f_{(r)}, [s']) = \sum_{p=0}^{r+1} \sum_{A=1}^{n} \tilde{f}_{A,p} \odot \nabla^p (s^A)' \tag{2.8}$$

where

$$\tilde{f}_{A,p} := (1 - \delta_{0,p})(J \otimes f_{A,p-1}) + (1 - \delta_{r+1,p}) \nabla f_{A,p} \tag{2.9}$$

As usual, in the above definition of $\tilde{f}_{A,p}$ we have assumed that $\delta_{0,p}$ and $\delta_{r+1,p}$ are the Kronecker deltas. Of course, given the contraction of $\tilde{f}_{A,p}$ with $\nabla^p(s^A)'$, generally some convention as to which of the $1 + 2(r_A + R_A + p)$ indices in a coordinate representation of $\tilde{f}_{A,p} \otimes \nabla^p(s^A)'$ are to be contracted must be followed when doing the contraction, but for equation (2.8) this convention is rather obvious and can be deduced from the requirement that $x \mapsto \tilde{f}_{A,p}(x) \odot (\nabla^p(s^A)')_x$ is a cross section of $T = \bigcup_{x \in X} T_x$, the tensor bundle of type (0, 1) over X. Thus, we will not dwell on the method of computing $\tilde{f}_{A,p} \odot \nabla^p(s^A)'$ here, referring the reader to the Appendix for more details.

Now, let us translate the result (2.8) into the other notation, using the definition

$$\tilde{f}_{(r+1)} := \bigoplus_{p=0}^{r+1} \left(\bigoplus_{A=1}^{n} \tilde{f}_{A,p} \right) \tag{2.10}$$

Just as in the case of equation (4.22) appearing in Banach and Piekarski (1997), this definition may be interpreted geometrically by saying that $x \mapsto \tilde{f}_{(r+1)}(x)$ is a cross section of the vector bundle

$$S^{\star}_{(r+1)} := \bigcup_{x \in X} S^{\star}_{(r+1),x} \tag{2.11}$$

where

$$S^{\star}_{(r+1),x} := L(S_{(r+1),x}, T_x) \tag{2.12}$$

But by equation (2.10) and the obvious formula

$$D^{r+1} s' = \bigoplus_{p=0}^{r+1} \left(\bigoplus_{A=1}^{n} \nabla^p (s^A)' \right) \tag{2.13}$$

an abbreviated expression for the covariant derivative of a scalar gauge-invariant variable of order r is

$$\nabla G(\cdot, f_{(r)}, [s']) = \langle \tilde{f}_{(r+1)}, D^{r+1} s' \rangle \tag{2.14}$$

and from the properties of $f_{(r)} \in \Gamma^{*}_{(r),F}$ we derive the useful identity

$$\langle \tilde{f}_{(r+1)}, D^{r+1}(\mathcal{L}_v s_b) \rangle = 0 \tag{2.15}$$

which holds for each C^{r+2} vector field v on X. Consequently, we may think of $\tilde{f}_{(r+1)}$ as being the element of $\Gamma^{\star}_{(r+1),F}$ (Banach and Piekarski, 1997, Section 4.3). *This observation completes our proof that* $\nabla G(\cdot, f_{(r)}, [s'])$ *is a gauge-invariant variable of order* $r + 1$.

Naturally, once the covariant vector field $\nabla G(\cdot, f_{(r)}, [s'])$ on X has been put into the canonical form (2.14), the remaining task is to explain the result of a multiple application of ∇ to $x \mapsto G(x, f_{(r)}, [s'])$. However, this result can be explained straightforwardly: the object $\nabla^p G(\cdot, f_{(r)}, [s'])$ is a gauge-invariant variable of order $r + p$. The above observations are quite universal, except for the assumption made explicitly at the beginning of this discussion that $G(\cdot, f_{(r)}, [s'])$ is a scalar field on X. Fortunately, even if $G(\cdot, f_{(r)}, [s'])$ is not such a scalar field on X [i.e., $G(\cdot, f_{(r)}, [s'])$ is a general gauge-invariant variable], the modified analysis proceeds in a way entirely similar to that already made familiar and the final conclusions remain basically intact, showing that $\nabla^p G(\cdot, f_{(r)}, [s'])$ is a gauge-invariant variable of order $r + p$.

As mentioned in Banach and Piekarski (1997, Section 4.3), any cross section of $\overline{F}_{(r)}$, not necessarily continuous, can in a sense be identified with the gauge-invariant variable of order r, and if only the vector bundle $\overline{F}_{(r)}$ does exist, as is quite often the case, then there are infinitely many cross sections $f_{(r)}$ of $\overline{F}_{(r)}$ and thus there are also infinitely many gauge-invariant variables of order r. With regard to the choice of an integer $r \gtrsim 0$ and a tensor bundle T in the definition of $\overline{F}_{(r)}$, this choice depends mostly on us, and different possible choices of r and T give rise to different gauge-invariant variables. Specifically, if we choose a fixed tensor bundle T, we are still free to introduce gauge-invariant variables of various orders.

Thus, taking the covariant derivative of $x \mapsto G(x, f_{(r)}, [s'])$ is not the only process for constructing a new gauge-invariant variable from an old one. For a fixed tensor bundle T, we also have the interesting possibility that the gauge-invariant variables

$$G_p := G(\cdot, f_{p(r_p)}, [s']), \qquad p = 1, 2, \ldots, k \tag{2.16}$$

of *generally* various orders, *which are cross sections of* T, can be multiplied by arbitrary, real-valued functions λ_p ($p = 1, 2, \ldots, k$) on X and then added:

$$G(x, f_{(r)}, [s']) := \sum_{p=1}^{k} \lambda_p(x) G(x, f_{p(r_p)}, [s']) \tag{2.17}$$

In these definitions, $\{r_1, r_2, \ldots, r_k\}$ is a sequence of integers ≥ 0, $r := \max(r_1, r_2, \ldots, r_k)$, and it may happen that some or all of these integers are identical. For essentially obvious reasons, the object $f_{p(r_p)}$ depends explicitly on p; this object plays the role previously played by $f_{(r)} \in \Gamma^{\star}_{(r),F}$ [see, e.g., equation (4.22) in Banach and Piekarski (1997)]. Hence, with the quantity

(2.17) formally defined as above, we easily find that there exists a cross section $x \mapsto f_{(r)}(x)$ of

$$S^{\star}_{(r)} = \bigcup_{x \in X} L(S_{(r),x}, T_x) \tag{2.18}$$

such that $f_{(r)} \in \Gamma^{\star}_{(r),F}$ and $G(x, f_{(r)}, [s'])$ can be expressed in the form

$$G(x, f_{(r)}, [s']) = \langle f_{(r)}(x), (D^r s')_x \rangle \tag{2.19}$$

Therefore, the quantity (2.17) is a gauge-invariant variable of order r.

This statement brings to an end our description of the basic processes for creating new gauge-invariant variables from old, the others being only combinations of them.

3. BASIC GAUGE-INVARIANT VARIABLES

3.1. The Role of a "Coordinate System" on Γ/Γ_L

In Banach and Piekarski (1997), we constructed the gauge-invariant variables in such a way that for each choice of $f_{(r)} \in \Gamma^{\star}_{(r),F}$, the object $G(\cdot, f_{(r)}, \cdot)$ defines a mapping from the quotient space Γ/Γ_L into a set of cross sections of T. Denoting this set by $\Gamma(T)$, we thus have

$$\Gamma/\Gamma_L \ni [s'] \mapsto G(\cdot, f_{(r)}, [s']) \in \Gamma(T) \tag{3.1}$$

where $G(\cdot, f_{(r)}, [s'])$ is an abbreviated notation for

$$X \ni x \mapsto G(x, f_{(r)}, [s']) \in T_x \tag{3.2}$$

If one should insist on the theory in which the quotient space Γ/Γ_L admits a "coordinate system" consisting of *finitely* many gauge-invariant variables

$$G(\cdot, f_{p(r_p)}, [s']) \in \Gamma(T_p), \qquad p = 1, 2, \ldots, l \tag{3.3}$$

it would of course be possible to find a *necessary* and *sufficient* condition that the equivalence classes of perturbations are *uniquely* determined from these *basic* variables. Can this be done within the formal structure of linear perturbation theory for an arbitrary system of covariant field equations?

In order to answer this question, we proceed as follows. Let $\{r_1, r_2, \ldots, r_l\}$ be a sequence of integers ≥ 0 and define $\bar{r}$ by $\bar{r} := \max(r_1, r_2, \ldots, r_l)$. We limit ourselves to the study of the case $\bar{r} \leq q$, where q is an integer which has exactly the same meaning as in equations (2.21) of Banach and

Piekarski (1997).[5] Assume further that to every gauge-invariant perturbation $[s'] \in \Gamma/\Gamma_L$ there is an associated set

$$\varphi([s']) := \{G_1(\cdot, [s']), G_2(\cdot, [s']), \dots, G_l(\cdot, [s'])\} \tag{3.4}$$

of gauge-invariant variables, characterized by the formulas

$$\begin{aligned} G_p(x, [s']) &:= G(x, f_{p(r_p)}, [s']) \\ &:= \langle f_{p(r_p)}(x), (D^{r_p}s')_x\rangle, \qquad p = 1, 2, \dots, l \end{aligned} \tag{3.5}$$

Geometrically, after denoting by T_p the tensor bundle over X (the details of the definition of T_p may depend on the particular integer p chosen), we can think of $f_{p(r_p)}$ as being the cross section of

$$S^{\star}_{p(r_p)} := \bigcup_{x\in X} L(S_{(r_p),x}, T_{p,x}) \tag{3.6}$$

where $T_{p,x}$ is the fiber of T_p through the point $x \in X$. As in Section 2, an identification of $G_p(\cdot, [s'])$ with the gauge-invariant variable of order r_p leads us to the conclusion that, for each C^{r_p+1} vector field v on $X, f_{p(r_p)}$ must satisfy the condition of the form

$$\langle f_{p(r_p)}, D^{r_p}(\mathcal{L}_v s_b)\rangle = 0, \qquad p = 1, 2, \dots, l \tag{3.7}$$

Given $f_{p(r_p)}$ and hence $G_p(\cdot, [s'])$, another postulated property of $\{G_p(\cdot, [s']); p = 1, 2, \dots, l\}$ is simply this: if $G_p(\cdot, [s'])$ is any one of the gauge-invariant variables appearing in the definition of $\varphi([s'])$ [see equations (3.4) and (3.5)], then $G_p(\cdot, [s'])$ cannot be obtained from $G_{p'}(\cdot, [s'])$, $p' \neq p$, through purely local (i.e., algebraic and differential) operations. Consequently, we are justified in saying that the set $\varphi([s'])$ is *linearly independent.*

After these preparations, we denote by Ω the set consisting of $\varphi([s'])$ for all $[s'] \in \Gamma/\Gamma_L$ and by ω, ω', and similar symbols the elements of Ω. A function φ from Γ/Γ_L onto Ω, defined by equation (3.4), is a linear map which assigns to each $[s'] \in \Gamma/\Gamma_L$ an element $\varphi([s']) \in \Omega$; thus Ω carries a canonical structure of a vector space induced by that of Γ/Γ_L. More precisely, Ω is a function space in which the usual operations of addition and scalar multiplication are introduced. A necessary and sufficient condition that φ be a one-to-one linear mapping of Γ/Γ_L onto Ω is that $\varphi([s'])$ equals a zero-vector of Ω if and only if $[s']$ equals a zero-vector of the quotient space Γ/Γ_L, i.e., if and only if $[s']$ can be identified with $[\mathcal{L}_v s_b]$, where v is an arbitrary vector field[6]

[5] As noted already (Banach and Piekarski, 1997, Section 2.2), since the objects H^I, $I = 1, 2, \dots, m$, depend only on s and its first q covariant derivatives $\nabla^p s$, and since they satisfy the condition $\sigma * H^I(\cdot, D^q s) = H^I[\cdot, D^q(\sigma * s)]$ for each I, it seems reasonable to refer to these equations as the covariant field equations of order q.

[6] Precisely speaking, v must be of class C^k (k sufficiently large); otherwise $\mathcal{L}_v s_b$ cannot be a classical solution of the linearized field equations.

on the "space-time" manifold X. It will be convenient to call this condition a *natural condition* for the existence of a "coordinate system" on Γ/Γ_L. Hence, motivated by these considerations, we obtain the following simple theorem.

Theorem 1. If the mapping φ: $\Gamma/\Gamma_L \to \Omega$ described above satisfies a *natural condition* for the existence of a "coordinate system" on Γ/Γ_L, then for each $\omega \in \Omega$ there is just one $[s'] \in \Gamma/\Gamma_L$ such that $\omega = \varphi([s'])$. In this case, the mapping φ: $\Gamma/\Gamma_L \to \Omega$ is said to be one-to-one and onto, and we can define the inverse of φ, φ^{-1}: $\Omega \to \Gamma/\Gamma_L$ by setting $(\varphi^{-1}\varphi)([s']) = [s']$).

Proof. Since the mapping φ: $\Gamma/\Gamma_L \to \Omega$ is linear, this theorem can be proved immediately. ■

The mapping φ: $\Gamma/\Gamma_L \to \Omega$ obeying the condition of Theorem 1 is fundamental for at least two reasons. First, $[s'] = \varphi^{-1}(\omega)$ is a gauge-invariant perturbation associated with $\omega \in \Omega$ (Banach and Piekarski, 1997, Section 4.1) and the objects appearing on the right-hand side of equation (3.4), *called the basic gauge-invariant variables*, are "coordinates" of $[s'] \in \Gamma/\Gamma_L$. Thus $[s']$ is uniquely determined from $\omega = \varphi([s'])$ and vice versa. This fact supports an interpretation of φ: $\Gamma/\Gamma_L \to \Omega$ as a "coordinate system" on Γ/Γ_L. Second, any gauge-invariant variable $G(\cdot, f_{(r)}, [s'])$ can be constructed *directly* from the basic variables $G_1(\cdot, [s'])$, $G_2(\cdot, [s'])$, . . . , $G_l(\cdot, [s'])$ through *purely* local operations. We discuss some aspects of this problem in Section 3.2.

Of course, at this level of generality little more than the definition, or the concept itself, is possible. In fact, as the theory presently stands, it is by no means clear what universal arguments we are to use in explicitly constructing the basic gauge-invariant variables and the one-to-one mapping of Γ/Γ_L onto Ω. Nevertheless, our primary task here was to show that the concept of basic gauge-invariant variables and the notion of a "coordinate system" on Γ/Γ_L can be discussed without ever presenting the detailed form of covariant field equations and without ever analyzing the specific properties of the background "space-time" geometry chosen. If one means by covariant field theories Einstein's gravity theory or other metric theories of gravity, one will be able to provide explicit and comparatively simple examples of the mapping φ: $\Gamma/\Gamma_L \to \Omega$. In a cosmological setting, these examples are primarily applicable to the case of an almost-Robertson–Walker universe model (Ellis and Bruni, 1989). Then, as explained already by Banach and Piekarski (1996a–c), the set $\varphi([s'])$ consists of 17 or 18 "geometrically" independent, not identically vanishing gauge-invariant variables. We can also obtain an analytical form of $\varphi([s'])$ for the infinitesimal perturbation of the metric tensor itself (pure gravity) defined on a fixed background de Sitter space-time (see Section 5).

3.2. Completeness of the Set $\varphi([s'])$ of Basic Gauge-Invariant Variables

Using the concepts of Section 3.1, we can now formulate our main theorem.

Theorem 2. Suppose that φ is a one-to-one mapping of Γ/Γ_L onto Ω defined as before; thus this mapping has exactly the same meaning as in Theorem 1. Let $\omega := \varphi([s'])$ be a set of basic gauge-invariant variables $G_p(\cdot, [s'])$, $p = 1, 2, \ldots, l$, associated with $[s'] \in \Gamma/\Gamma_L$ [see equation (3.4)]. Under these circumstances, a knowledge of ω is all one needs in the sense that if x represents an arbitrary point of X and $G(\cdot, f_{(r)}, [s'])$ is any gauge-invariant variable of order r, then the value of $G(\cdot, f_{(r)}, [s'])$ at $x \in X$ is uniquely specified by giving the germs (Choquet-Bruhat *et al.*, 1989) of basic gauge-invariant variables $G_p(\cdot, [s'])$, $p = 1, 2, \ldots, l$, at $x \in X$.

Remark 1. Alternatively put, this theorem means that the set $\omega := \varphi([s'])$, which is finite and linearly independent (in the sense of Section 3.1), forms a *complete set* of basic gauge-invariant variables.

Remark 2. A necessary and sufficient condition for the set $\omega := \varphi([s'])$ to be a complete set for each choice of $[s'] \in \Gamma/\Gamma_L$ is automatically assured by our definition of the mapping φ: $\Gamma/\Gamma_L \to \Omega$ and is exactly the same as the *natural condition* for the existence of a "coordinate system" on Γ/Γ_L (see Section 3.1).

Proof. According to the investigations in Banach and Piekarski (1997, Section 4.3), the gauge-invariant variable $G(\cdot, f_{(r)}, [s'])$ coincides with the mapping

$$X \ni x \mapsto \langle f_{(r)}(x), (D^r \bar{s}')_x \rangle \in T_x \tag{3.8}$$

where $\bar{s}'$ is an arbitrary member of $[s'] \in \Gamma/\Gamma_L$ and where $x \mapsto f_{(r)}(x)$ is a cross section, not necessarily continuous, of the vector subbundle $\overline{F}_{(r)}$ of $S^{\star}_{(r)}$. Clearly, since $G(x, f_{(r)}, [s'])$ is characterized by

$$G(x, f_{(r)}, [s']) := \langle f_{(r)}(x), (D^r \bar{s}')_x \rangle \tag{3.9}$$

the form of a quantity appearing on the right-hand side of equation (3.9) is completely independent of the choice of $\bar{s}' \in [s']$ and this conclusion holds for each $x \in X$. Thus, combining equations (3.8) and (3.9), we see that $[s'] \mapsto G(x, f_{(r)}, [s'])$ defines a mapping of Γ/Γ_L into T_x. Together with the identification rule $[s'] = \varphi^{-1}(\omega)$ of Theorem 1, if we let ω denote the set of basic gauge-invariant variables $G_p(\cdot, [s'])$, $p = 1, 2, \ldots, l$, associated with $[s'] \in \Gamma/\Gamma_L$, the typical (i.e., analytical) expression of the aforementioned fact is

$$G(x, f_{(r)}, [s']) = \tilde{G}(x, f_{(r)}, \omega) \tag{3.10}$$

where

$$\tilde{G}(x, f_{(r)}, \cdot) := G(x, f_{(r)}\ \varphi^{-1}(\cdot)) \tag{3.11}$$

Here $\tilde{G}(x, f_{(r)}, \cdot)\colon \Omega \to T_x$ represents a linear functional, that is, a linear function whose arguments are basic gauge-invariant variables $\omega' \in \Omega$ associated with various elements of Γ/Γ_L. But from equation (3.9) it follows that our constructions are local, because the original gauge-dependent variable $\bar{s}'$ is allowed to enter the definition of $\langle f_{(r)}(x), (D^r \bar{s}')_x \rangle$ only through $\bar{s}'(x)$ and its covariant derivatives $(\nabla^p \bar{s}')_x$ up to order r. As a consequence, all the observations and all the results of this paper, especially Theorem 1, remain valid, *mutatis mutandis*, when we replace X by any open set $\vartheta \subset X$ in the statements and proofs. Therefore, after denoting, respectively, by $s'_{|\vartheta}$ and $G_{p|\vartheta}$ the restrictions of s' and $G_p(\cdot, [s'])$ to ϑ and by $\omega_{|\vartheta}$ the set of basic gauge-invariant variables $G_{p|\vartheta}$, $p = 1, 2, \ldots, l$, associated with the equivalence class $[s'_{|\vartheta}]$ of $s'_{|\vartheta}$, we immediately find that if x belongs to ϑ, the object $G(x, f_{(r)}, [s'])$ depends in essence on $[s'_{|\vartheta}]$; in other words, we have

$$G(x, f_{(r)}, [s']) = \overline{G}_{\vartheta}(x, f_{(r)}, [s'_{|\vartheta}]), \qquad x \in \vartheta \tag{3.12}$$

However, $[s'_{|\vartheta}]$ is uniquely determined from $\omega_{|\vartheta}$ (just as $[s']$ is uniquely determined from ω), and thus for each choice of $x \in \vartheta$ and $f_{(r)} \in \Gamma^{\star}_{(r),F}$ we can regard $G(x, f_{(r)}, [s'])$ as a function of $\omega_{|\vartheta}$:

$$G(x, f_{(r)}, [s']) = \tilde{G}_{\vartheta}(x, f_{(r)}, \omega_{|\vartheta}), \qquad x \in \vartheta \tag{3.13}$$

Obviously, equation (3.13) is valid for all open subsets ϑ of X containing $x \in X$, *however small*. Then a standard argument of differential geometry yields the conclusion that the value of $G(\cdot, f_{(r)}, [s'])$ at $x \in X$ is uniquely specified by giving the germs of basic gauge-invariant variables $G_p(\cdot, [s'])$, $p = 1, 2, \ldots, l$, at $x \in X$. *These germs will be denoted collectively by* ω_x. With such a convention in mind, we finally observe that there exists a function $\overline{G}(x, f_{(r)}, \omega_x)$ related to $G(x, f_{(r)}, [s'])$ by the equation

$$\overline{G}(x, f_{(r)}, \omega_x) = G(x, f_{(r)}, [s']), \qquad x \in X \tag{3.14}$$

and a derivation of this local equation completes the proof of Theorem 2. ∎

If now, instead of directly using equation (3.14), we relate the infinitesimal perturbation s' of s_b to the tensor fields $(s^A)'$ as in the formula (2.2), we get from the definition (3.9) and the analysis of Section 2 an explicit expression for $G(x, f_{(r)}, [s'])$ which tells us that $G(x, f_{(r)}, [s'])$ is a linear combination of $(s^A)'(x)$ and the first r covariant derivatives of $(s^A)'$ evaluated at $x \in X$ [see, e.g., equation (2.5)]. Then in such a combination we can try to substitute the similar formulas for $G_p(x, [s'])$ that result from exploiting the definitions (3.5) of basic gauge-invariant variables, so eventually obtaining a *less abstract* realization of equation (3.14). The following hypothesis should help elucidate the precise meaning of this conjecture.

Hypothesis. For sufficiently smooth gauge-invariant variables $G(\cdot, f_{(r)}, [s'])$ and $G_p(\cdot, [s'])$, $p = 1, 2, \ldots, l$, Theorem 2 implies a stronger one which is more immediately useful: any $G(\cdot, f_{(r)}, [s'])$ is obtainable linearly from the basic gauge-invariant variables $G_p(\cdot, [s'])$, $p = 1, 2, \ldots, l$, through purely algebraic and differential operations.

Remark. Usually, the term "smooth" means C^∞, but here is used in preference to C^∞, because in fact we do not require "infinite" smoothness.

In a series of recent papers (Banach and Piekarski, 1996a–c), it was explicitly demonstrated that one concrete setting where one is led to the *full confirmation* of the above hypothesis occurs when considering linear perturbation theory for Einstein's field equations or the Einstein–Liouville coupled system of equations. Our concern there was with the complete, finite set of basic gauge-invariant variables as applied to cosmology (Ryan and Shepley, 1975). To model the real universe in a mathematically tractable structure, emphasis was placed on discussing the simplest case in which the background space-time geometry is that of a $k = 0$ or $k \pm 1$ Robertson–Walker space-time (Kramer *et al.*, 1980). However, the extension of our previous results to other space-time geometries [e.g., the Bianchi type I background (Hawking and Ellis, 1973)] is very straightforward. Of course, finding a method to solve the problems offered by these cosmological models is no proof by itself of the universality of the method. To defend the present approach on still better grounds, we should attempt to show in addition that a confirmation of the hypothesis in important but particular cases is just an illustration of how the general theory works when no information about the symmetry properties of the background is given. Unfortunately, this problem is extremely difficult and thus its more satisfactory solution is clearly beyond the scope of the formalism developed here and in Banach and Piekarski (1996a–c).

4. LINEAR PERTURBATION EQUATIONS FOR THE BASIC GAUGE-INVARIANT VARIABLES

As can be seen from the discussion in Section 3.2 of Banach and Piekarski (1997), the system of linear differential equations for the determination of s' is given by

$$\langle H^I_{(q)}, D^q s' \rangle = 0, \qquad I = 1, 2, \ldots, m \tag{4.1}$$

Recall, when using this system, that the mapping

$$X \ni x \mapsto \langle H^I_{(q)}(x), (D^q s')_x \rangle \in V^I_x \tag{4.2}$$

which is a cross section of

$$V^I := \bigcup_{x \in X} V^I_x \tag{4.3}$$

may be interpreted as a gauge-invariant variable of order q. Thus, if s' and v are, respectively, an arbitrary C^q cross section of S and an arbitrary C^{q+1} vector field on X, then the following condition is automatically obeyed everywhere:

$$\langle H^I_{(q)}, D^q s' \rangle = \langle H^I_{(q)}, D^q(s' + \mathcal{L}_v s_b) \rangle \tag{4.4}$$

However, given the results of Section 3, Theorem 2 enables us to show that, for all possible choices of the pair (I, x), there exists a linear "function" $\overline{G}^I(x, \cdot)$: $\Omega_x \to V^I_x$ defined on the space Ω_x of germs $\omega_x \in \Omega_x$ of basic gauge-invariant variables ω at $x \in X$ such that

$$\langle H^I_{(q)}(x), (D^q s')_x \rangle = \overline{G}^I(x, \omega_x) \tag{4.5}$$

If, in line with the more concrete interpretation of Theorem 2 being developed (see the formulation of our hypothesis), any gauge-invariant variable is obtainable locally from the basic gauge-invariant variables

$$G_p := G_p(\cdot, [s']), \qquad p = 1, 2, \ldots, l \tag{4.6}$$

through purely algebraic and differential operations, equation (4.5) must be understood as

$$\langle H^I_{(q)}(x), (D^q s')_x \rangle = \sum_{k=0}^{q} \sum_{p=1}^{l} h^{I,p}_k(x) \odot (\nabla^k G_p)_x \tag{4.7}$$

where the objects $h^{I,p}_k(x)$ are tensorial coefficients which depend on x through a background solution $s_b(x)$ to the nonlinear field equations. Moreover, explaining the meaning of the symbol $\odot$, this symbol indicates that $h^{I,p}_k(x) \odot (\nabla^k G_p)_x \in V^I_x$ is a value of $h^{I,p}_k(x)$ on $(\nabla^k G_p)_x$, i.e., a contraction of $h^{I,p}_k(x)$ with $(\nabla^k G_p)_x$.

The formula (4.7) carries with it an interesting implication. As noted in Banach and Piekarski (1997, Section 4.1), we may think of Γ_C/Γ_L as being the subspace of Γ/Γ_L.[7] Consequently, if

$$\omega := \{G_1, G_2, \ldots, G_l\} \tag{4.8}$$

belongs to $\Omega_C := \varphi(\Gamma_C/\Gamma_L)$, the image space of Γ_C/Γ_L under φ, we find that

[7] By definition, the elements of Γ_C are classical solutions s' of equations (4.1).

the set $\omega \in \Omega_C$ of basic gauge-invariant variables is constrained to satisfy the following system of covariant differential equations:

$$\sum_{k=0}^{q} \sum_{p=1}^{l} h_k^{I,p} \odot \nabla^k G_p = 0, \qquad I = 1, 2, \ldots, m \tag{4.9a}$$

Because of the existence of this system, it should now be clear what equations (4.1) really are: these are basic gauge-invariant equations which may be reexpressed in a manifestly gauge-invariant form (4.9a).

Starting with the linearized field equations for Einstein's gravity theory applied to an almost-Robertson–Walker universe model (Ellis and Bruni, 1989), it will be possible to provide nontrivial examples of equations (4.1) and (4.9a). To put this statement more succinctly in terms of concrete formulas, we mention that equations (3.12a)–(3.12e) and (4.18a)–(4.18e) considered in one of our previous papers (Banach and Piekarski, 1996b, pp. 282 and 293) may be taken as such examples. Another example is given below in Section 5.

At this stage of the analysis, we say that the system (4.9a) gives usually an *underdetermined* system of differential equations for the specification of basic gauge-invariant variables ω. A determinate system results only if we derive some additional "constraint" equations for ω. In our somewhat symbolic notation, these additional equations can be written as

$$\sum_{0 \le k \le r_I} \sum_{p=1}^{l} h_k^{I,p} \odot \nabla^k G_p = 0 \tag{4.9b}$$

where

$$I = m + 1, m + 2, \ldots, l' \tag{4.9c}$$

with r_I and l' denoting the appropriate integers.

When examining the covariant field theories of physical interest (Banach and Piekarski, 1996a–c), we shall always find an explicit representation for the tensorial coefficients $h_k^{I,p}$ ($m + 1 \le I \le l'$) and equations (4.9b). As an illustration of this fact, see equations (4.18f) and (4.18g) in Banach and Piekarski (1996b, pp. 293 and 294). Also, in each particular case, it will be "easy" to prove that (i) equations (4.9a) and (4.9b) form a closed set of partial differential equations for the determination of ω and that (ii) every classical solution ω to these equations is an element of Ω_C, the image space of Γ_C/Γ_L under φ. This is crucial: this establishes one possible sense in which the present formalism determines potentially everything, namely that one can extract $[s'] \in \Gamma_C/\Gamma_L$ from $\omega := \varphi([s']) \in \Omega_C$ in a unique way.

5. APPLICATION: EINSTEIN'S GRAVITY THEORY AND A FIXED BACKGROUND DE SITTER SPACE-TIME

5.1. Preliminaries

In this section, we shall study the perturbation method on the basis of Einstein's general theory of relativity. For an empty space ($T_{ab} = 0$), the metric g_{ab} defined on X (dim $X = 4$) is assumed to obey the covariant field equations given by

$$R_{ab} - \tfrac{1}{2}Rg_{ab} + \Lambda g_{ab} = 0 \tag{5.1}$$

in the standard notation ($c = 8\pi G = 1$); the space-time metric g_{ab} has signature $(-, +, +, +)$. We introduce a *fixed background metric* γ_{ab}, which is the space-time metric of constant curvature K ($K = \frac{1}{3}\Lambda$). The resulting background space for $\Lambda > 0$ is *de Sitter space-time* (Hawking and Ellis, 1973, p. 124). It will be convenient to take the contravariant forms g^{ab} and γ^{ab} of the metrics g_{ab} and γ_{ab} to be more fundamental and the covariant forms g_{ab} and γ_{ab} as derived from them by the relations

$$g^{ab}g_{bc} = \gamma^{a}{}_{c} = \gamma_{c}{}^{a} = \gamma^{ab}\gamma_{bc} \tag{5.2}$$

where $\gamma^{a}{}_{c} = \gamma_{c}{}^{a}$ is the Kronecker delta ($\gamma^{a}{}_{c} := \delta^{a}{}_{c}$). Here and throughout this section, we adopt the *summation convention* whereby a repeated index implies summation over all values of that index.

If g^{ab} depends differentiably on $\epsilon \in \mathcal{U}$ (Banach and Piekarski, 1997, Section 3.1), it will be possible to define the infinitesimal perturbation q^{ab} of γ^{ab} as follows:

$$q^{ab} := \left(\frac{\partial g^{ab}}{\partial \epsilon}\right)_{\epsilon=0} \tag{5.3}$$

Let a slash denote the covariant derivative of a tensor field $B^{ab\cdots c}$ with respect to γ_{ab} [$B^{ab\cdots c}{}_{|d} := \nabla_d B^{ab\cdots c}$; here the symbol ∇_d has the same meaning as in Wald (1984, pp. 30–36)]. A careful analysis of

$$\left[\frac{\partial}{\partial \epsilon}\left(R^{a}{}_{b} - \frac{1}{2}R\gamma^{a}{}_{b} + \Lambda\gamma^{a}{}_{b}\right)\right]_{\epsilon=0} = 0 \tag{5.4}$$

then shows that the linear differential equations for q^{ab} are given by

$$\gamma_{ce}(\gamma_{bd}G^{acde} - \tfrac{1}{2}\gamma^{a}{}_{b}\gamma_{df}G^{cdef}) = 0 \tag{5.5}$$

where

$$G^{abcd} := -\frac{\Lambda}{3}\, q^{e[a}(\gamma^{b][c}\gamma^{d}{}_{e} - \gamma^{b]d}\gamma^{c}{}_{e}) - \gamma^{df}\gamma^{e[a}q^{b]c}{}_{|ef} + \gamma^{cf}\gamma^{e[a}q^{b]d}{}_{|ef} \tag{5.6}$$

and where the process of alternation over two upper indices a and b in the expression on the right-hand side of the above formula is denoted by square brackets (Schouten, 1954). Obviously, combination of (5.5) and (5.6) yields the desired system of equations which describes the "evolution" in space-time of q^{ab}. Another role of the system (5.5) is in providing an example of equations (4.9a). The validity of this statement follows mostly from the fact that, as will be demonstrated in Section 5.2, the tensor field G^{abcd} is a gauge-invariant variable of order 2 and thus the system (5.5) is a manifestly gauge-invariant form of equations (5.4).

Consider the local coordinate system (x^a) in a neighborhood $\mathcal{N}_x$ of $x \in X$ with four functions x^a: $\mathcal{N}_x \to \mathbb{R}$ $(a = 1, 2, \ldots, 4)$ whose values at $x' \in \mathcal{N}_x$ are the coordinates of the point x' of the space-time manifold X. The directional derivatives along the coordinate lines at $x' \in \mathcal{N}_x$ form a basis of an N-dimensional vector space $(N = 4)$:

$$e_a(x') := (\partial/\partial x^a)_{|x'} \tag{5.7}$$

This space, called the tangent space $T_{x'}(X)$, consists of the tangent vectors at x'. The basis $\{e_a(x')\}$ is called a *coordinate basis* or *holonomic frame*. For each $x' \in X$ in a coordinate domain, the four linearly independent 1-forms $e^a(x')$, which are uniquely determined by

$$e^a(x') \odot e_b(x') = \gamma^a{}_b \tag{5.8}$$

form a basis $\{e^a(x')\}$ of the *dual space* $T^*_{x'}(X)$ of the tangent space $T_{x'}(X)$. This basis $\{e^a(x')\}$ is said to be dual to the basis $\{e_a(x')\}$ of $T_{x'}(X)$.

Now, using the terminology and notation of Banach and Piekarski [(1997), equations (3.3) and (3.5)], the explicit formulas or interpretations for n, A, s_b, s^A_b, $s' = \nabla^0 s'$, $(s^A)' = \nabla^0(s^A)'$, and so forth are easily deduced from the above considerations as follows:

$$n = A = 1 \tag{5.9a}$$

$$s_b = s^A_b = s^1_b = \gamma^{ef} e_e \otimes e_f \tag{5.9b}$$

$$s' = (s^A)' = (s^1)' = q^{ef} e_e \otimes e_f \tag{5.9c}$$

$$\nabla s' = \nabla^1 s' = q^{ef}{}_{|g} e^g \otimes e_e \otimes e_f \tag{5.9d}$$

$$\nabla^2 s' = q^{ef}{}_{|gh} e^h \otimes e^g \otimes e_e \otimes e_f \tag{5.9e}$$

For brevity, let us set $\lambda := (abcd)$ and then define the objects $f^{(\lambda)}_p := f^{(\lambda)}_{A,p} = f^{(\lambda)}_{1,p}$ $(p = 0, 1, 2)$ dual to $\nabla^p s' = \nabla^p(s^A)' = \nabla^p(s^1)'$ $(p = 0, 1, 2)$ by

$$f^{(\lambda)}_0 := -\frac{2\Lambda}{3} \gamma_g{}^{[a}\gamma^{b][c}\gamma^{d]}{}_h e^g \otimes e^h \tag{5.10a}$$

$$f^{(\lambda)}_1 := 0 \tag{5.10b}$$

$$f^{(\lambda)}_2 := -2\gamma^{e[a}\gamma^{b]}{}_h \gamma_g{}^{[c}\gamma^{d]f} e_f \otimes e_e \otimes e^g \otimes e^h \tag{5.10c}$$

Before we proceed to the alternative expression for G^{abcd}, it is necessary to note that

$$(e^a \otimes e^b) \odot (e_c \otimes e_d) = \gamma^a{}_c \gamma^b{}_d \tag{5.11}$$

and that a similar contraction of $e_a \otimes e_b \otimes e^c \otimes e^d$ with $e^e \otimes e^f \otimes e_g \otimes e_h$ yields $\gamma_a{}^e \gamma_b{}^f \gamma^c{}_g \gamma^d{}_h$. By virtue of these conventions the quantity G^{abcd} may be written in a more compact form:

$$G^{abcd} = \sum_{p=0}^{2} f_p^{(\lambda)} \odot \nabla^p s' \tag{5.12}$$

Inasmuch as

$$f_{\{2\}}^{(\lambda)} := \bigoplus_{p=0}^{2} f_p^{(\lambda)} \tag{5.13}$$

is, for each λ, a cross section of the vector subbundle $F_{(2)}$ of $S^*_{(2)}$ (see Section 5.2), this result for G^{abcd} agrees exactly with equation (2.5) and thus defines a scalar gauge-invariant variable of order 2:

$$G^{abcd} = \langle f_{\{2\}}^{(\lambda)}, D^2 s' \rangle \tag{5.14}$$

Clearly, the aforementioned concepts are tied to the choice of a particular holonomic frame $\{e_a\}$, and the gauge-invariant quantity which does not depend on this choice is given by

$$G = G^{abcd} e_a \otimes e_b \otimes e_c \otimes e_d \tag{5.15}$$

We finally mention the following. In general, we have $\Lambda \neq 0$ and our constructions above are valid when $\Lambda > 0$ or $\Lambda = 0$. A particular case is that in which $\Lambda = 0$; in that case (X, γ_{ab}) is the simplest empty space-time in general relativity, namely, Minkowski space-time.

5.2. Discussion of the Meaning of G^{abcd}

Let $\{B(\epsilon, x); \epsilon \in \mathcal{U}\}$ be a curve of geometrical objects obtainable tensor-algebraically from the metrics $g^{ab}(\epsilon', x)$, $\epsilon' \in \mathcal{U}$, and their first-, second-, or higher order covariant derivatives with respect to $\gamma_{ab}(x)$, and suppose that $B(\epsilon, x)$ depends differentiably on ϵ. With the abbreviation $B_0 := (B)_{\epsilon=0}$, we can define the quantity δB which represents the "first variation" of B_0:

$$\delta B := \left(\frac{\partial B}{\partial \epsilon} \right)_{\epsilon=0} \tag{5.16}$$

As shown already by Ehlers (1973), this first variation is invariant under the action of an "infinitesimal diffeomorphism" (Wald, 1984) if and only if the following condition holds for each vector field v on X:

$$\mathcal{L}_v B_0 = 0 \tag{5.17}$$

In order to satisfy the above equation, it is necessary to use a scalar B that is constant in the unperturbed space-time (X, γ_{ab}), or any tensor $B^{ab}{}_{\dots cd}$ that vanishes in (X, γ_{ab}), or a tensor whose "background value" is a constant linear combination of products of Kronecker deltas $\gamma^a{}_b$ (Stewart and Walker, 1974; Stewart, 1990).

Apparently, what seems unsatisfactory about the gauge-invariant variables G^{abcd} introduced in Section 5.1 is that although they are gauge-invariant objects of order 2 by construction,[8] it is not clear *a priori* to which quantity B they correspond. However, if R_{abcd} is the Riemann tensor of the connection defined by g_{ab} and γ_{ab} is the unperturbed space-time metric of constant curvature K ($K = \frac{1}{3}\Lambda \geq 0$), we immediately find from

$$R^{ab}{}_{cd} := g^{ae}g^{bf}R_{efcd} \tag{5.18}$$

and

$$(R^{ab}{}_{cd})_{\epsilon=0} = \frac{\Lambda}{3}\,(\gamma^a{}_c\gamma^b{}_d - \gamma^a{}_d\gamma^b{}_c) \tag{5.19}$$

that

$$\mathcal{L}_v[(R^{ab}{}_{cd})_{\epsilon=0}e_a \otimes e_b \otimes e^c \otimes e^d] = 0 \tag{5.20}$$

Thus

$$\delta R^{ab}{}_{cd} := \left(\frac{\partial R^{ab}{}_{cd}}{\partial \epsilon}\right)_{\epsilon=0} \tag{5.21}$$

is a gauge-invariant quantity with a simple geometric and physical meaning, and we can express G^{abcd} in terms of $\delta R^{ab}{}_{cd}$:

$$G^{abcd} = (\delta R^{ab}{}_{ef})\gamma^{ec}\gamma^{fd} \tag{5.22}$$

This way, the internal consistency of the formalism and the gauge-invariant character of G^{abcd} are demonstrated and understood from still another viewpoint.

With the same notation as in Banach and Piekarski [(1997), equation (2.16b)] and Section 5.1, we now define $S^*_{(2)}$ by

$$S^*_{(2)} := \bigoplus_{p=0}^{2} S^*_p \tag{5.23}$$

[8] The components G^{cdef} of G with respect to an arbitrary coordinate basis are gauge-invariant, because we get $G^{cdef} = 0$ when $q^{cd} = (\mathcal{L}_v s_b)^{cd}$. Concerning the definition of s_b, see equation (5.9b).

with S_p^* denoting the dual of the tensor bundle S_p of type $(2, p)$ over X (Dieudonné, 1972, p. 119). Applying the general construction of the vector subbundle $F_{(r)}$ of $S_{(r)}^*$ to the particular case where r equals 2 and $S_{(2)}^*$ is given by (5.23) (Banach and Piekarski, 1997, Section 4.2), we finally deduce from equations (5.10)–(5.14) that since G^{abcd} is a scalar gauge-invariant variable of order 2, the object $f_{(2)}^{(\lambda)}$ appearing on the right-hand side of equation (5.14) can be regarded as a cross section of $F_{(2)}$. Because of this crucial fact, the specific calculations we present here are *indeed* an example of the general formalism developed within the framework of linear perturbation theory for an arbitrary system of covariant field equations.

5.3. Relation of $\{G^{abcd}\}$ to the Complete Set of Basic Gauge-Invariant Variables

With these preparations, we are now ready to prove the following theorem.

Theorem 3. Let

$$s_b := \gamma^{cd} e_c \otimes e_d \tag{5.24}$$

be a background solution of Einstein's field equations for empty space (i.e., the contravariant space-time metric of constant curvature $\frac{1}{3}\Lambda$ described in Section 5.1), and let $[s']$ denote the equivalence class of

$$s' := q^{cd} e_c \otimes e_d \tag{5.25}$$

Define $G^{cdef}(\cdot, [s'])$ by equation (5.6), and suppose that $\varphi([s'])$ is related to the set $\{G^{cdef}(\cdot, [s'])\}$ by

$$\varphi([s']) := \{G^{cdef}(\cdot, [s'])\} \tag{5.26}$$

where $[s'] \in \Gamma/\Gamma_L$. Under these circumstances, if $\varphi([s'])$ is a zero-vector of the space Ω consisting of $\varphi([\bar{s}'])$ for all $[\bar{s}'] \in \Gamma/\Gamma_L$, then there exists a vector field v on the space-time manifold X such that $[s']$ for which $\varphi([s']) = 0$ is the equivalence class of $\mathcal{L}_v s_b$.

Remark 1. Interpreting this theorem, the mapping $\varphi: \Gamma/\Gamma_L \to \Omega$ defined by

$$\Gamma/\Gamma_L \ni [s'] \mapsto \varphi([s']) \in \Omega \tag{5.27}$$

satisfies a *natural condition* for the existence of a "coordinate system" on Γ/Γ_L.

Remark 2. Since $\varphi([s'])$ equals a zero-vector of Ω if and only if $[s']$ equals $[\mathcal{L}_v s_b]$ for some v, we can think of $\omega := \varphi([s'])$ as being the complete set of basic gauge-invariant variables (in the sense of Section 3.2).

Proof. If the equivalence class $[s']$ of $s' := q^{cd}e_c \otimes e_d$ happens to satisfy the condition $\varphi([s']) = 0$, we easily find from the definition (5.26) that

$$\frac{\Lambda}{3} q^{e[a}(\gamma^{b]c}\gamma^{d}{}_{e} - \gamma^{b]d}\gamma^{c}{}_{e}) + \gamma^{df}\gamma^{e[a}q^{b][c}{}_{|ef} - \gamma^{cf}\gamma^{e[a}q^{b]d}{}_{|ef} = 0 \tag{5.28}$$

However, this constraint is a *necessary* and *sufficient* condition [Truesdell and Toupin, 1960, equation (84.12), p. 352] that, given a symmetric tensor field $s' := q^{cd}e_c \otimes e_d$ on X, there exists a vector field v on X such that $s' = \mathcal{L}_v s_b$. Hence we have $[s'] = [\mathcal{L}_v s_b]$, and the proof of Theorem 3 is complete. ■

5.4. Knowledge About $[s'] \in \Gamma_C/\Gamma_L$ by Means of the Covariant Field Equations for Basic Gauge-Invariant Variables

A complete set of symmetry properties for G^{abcd} is $G^{abcd} = G^{[ab][cd]}$ and $G^{a[bcd]} = 0$; thus there are 20 linearly independent, not identically vanishing components in $\{G^{abcd}\}$. It then follows that since the system (5.5) consists of only 10 algebraically independent equations, these equations do not form a determinate system of equations for the specification of $\{G^{abcd}\}$. However, the definition (5.6) of G^{abcd} implies

$$\gamma_{ce}\gamma_{df}G^{abef}{}_{|g} + \gamma_{ge}\gamma_{cf}G^{abef}{}_{|d} + \gamma_{de}\gamma_{gf}G^{abef}{}_{|c} = 0 \tag{5.29}$$

and equations (5.5) and (5.29) represent an explicit, closed system of covariant field equations for the determination of $\{G^{abcd}\}$. In this way, the "constraint" equations (5.29), which are an example of the system (4.9b), may be added to equations (5.5) without the necessity of using the definition (5.6) [i.e., without the need of expressing G^{abcd} in terms of q^{ab}]

Let us ask now to what extent a knowledge of the classical solutions of equations (5.5) and (5.29) determines the equivalence classes of perturbations. This problem can be studied if we prove the following theorem.

Theorem 4. Every C^1 solution of equations (5.5) and (5.29) for $\{G^{abcd}\}$ belongs to Ω_C, the image space of Γ_C/Γ_L under φ.

Remark. Such a proof requires very careful examination if a sound and consistent development is to be achieved for the theory of perturbations at the level of Einstein's field equations (5.1) and the background space-time metric γ_{ab} of constant curvature $\frac{1}{3}\Lambda$. It must be stressed that as the theory presently stands, the validity of Theorem 4 is not evident, and we would like to show that the solution of equations (5.5) and (5.6) for q^{ab} is equivalent to solving equations (5.5) and (5.29) for G^{abcd}. The gist of the point made by Theorems 3 and 4 is that the information content contained in the gauge-invariant perturbation does not contract as the level of description passes from $[s'] = [q^{ab}e_a \otimes e_b] \in \Gamma_C/\Gamma_L$ to $\varphi([s']) = \{G^{abcd}(\cdot, [s'])\} \in \Omega_C$, since the passage essentially involves a complete set of basic gauge-invariant variables and every classical solution of equations (5.5) and (5.29) determines the equivalence class of perturbations.

Proof. Suppose that $\{G^{abcd}\}$ is a classical solution of equations (5.5) and (5.29). By means of arguments which are formally in the same form as those given in Trautman (1962) and Pirani (1965), it is then possible, provided G^{abcd} is of class C^1, to deduce from equations (5.29) and the symmetry properties of G^{abcd} the existence of a symmetric tensor field $s' := q^{ab}e_a \otimes e_b$ on X satisfying equations (5.6). On substitution of (5.6) for G^{abcd} into equations (5.5), the object $\{q^{ab}\}$ obtainable from the components of the aforementioned tensor field can be regarded as a classical solution of equations (5.5) and (5.6). Thus $\{G^{abcd}\}$ belongs to Ω_C, the image space of Γ_C/Γ_L under φ,[9] and this observation completes the proof of Theorem 4. ∎

6. FINAL REMARKS

In this and the companion paper (Banach and Piekarski, 1997), we have formulated linear perturbation theory for an arbitrary system of covariant field equations in such a way that the notion of a complete set of basic gauge-invariant variables is structurally universal, i.e., it holds regardless of the precise forms of s_b and $H^l_{(q)}$. No doubt, it is useful to have a universal formalism for the description of the equivalence classes of perturbations, as far as the basic structure of the theory is concerned. The condition (2.24) given in Banach and Piekarski (1997) is universal, and there is little reason to believe that such a universality should suddenly disappear as the full nonlinear equations $H^l(\cdot, D^q s) = 0$ for s are approximated by $H^l(\cdot, D^q s_b) = 0$ and $\langle H^l_{(q)}, D^q s'\rangle = 0$. This aspect of the gauge problem was investigated in various directions in this work. Beginning from Einstein's gravity theory, the new concepts developed were successfully applied to the construction of $\varphi: \Gamma/\Gamma_L \to \Omega$ for the cases of a fixed background de Sitter space-time (see Section 5) and an almost-Robertson–Walker universe model (Ellis and Bruni, 1989; Banach and Piekarski, 1996a–c). In addition, we have already verified that nontrivial extensions to homogeneous but anisotropic cosmological models (Ryan and Shepley, 1975) are also possible. However, since these extensions (i.e., the explicit definition of *finitely many* basic gauge-invariant variables and the explicit construction of a coordinate system on Γ/Γ_L) are not immediate, they will be treated in a separate paper. The applications

[9] As noted already, combination of (5.5) and (5.6) yields the desired equations for q^{ab}, and $s' := q^{ab}e_a \otimes e_b$ is an element of Γ_C if and only if the components q^{ab} of s' satisfy these equations.

made, though still not numerous, indicate the usefulness of the ideas presented, and hold promise for their applicability to problems which competitive theories have not been able to treat adequately.

Among the issues that can be studied systematically with this sort of approach, the examination of the effect of using a *semiclassical description* in which the background geometry is taken in the classical framework and the gauge-invariant perturbations are considered as *quantum variables* presents a most interesting challenge. Clearly, there are many ways of performing this task, and a very natural way consists in applying the methods of symplectic geometry and geometric quantization (Woodhouse, 1991). For a Lagrangian formulation of covariant field theories (Lee and Wald, 1990), the important object is a "presymplectic form" $s_b \mapsto \theta(s_b|\cdot, \cdot)$ defined on the space Γ of cross sections s_b of the vector bundle S. Such a presymplectic form can be used to construct a real-valued, bilinear functional of two "infinitesimal perturbations" s' and $\bar{s}'$ of s_b, denoted $\theta\ (s_b|s', \bar{s}')$. This functional satisfies the property that when s_b is a solution to the nonlinear field equations and s' and $\bar{s}'$ solve the linearized field equations, then $\theta(s_b|s', \bar{s}')$ is gauge-invariant, i.e., we have

$$\theta(s_b|s' + \mathcal{L}_v s_b, \bar{s}' + \mathcal{L}_{\bar{v}} s_b) = \theta(s_b|s', \bar{s}') \tag{6.1}$$

where v and $\bar{v}$ are two arbitrary vector fields on X. In other words, the two-form $\theta(\cdot|\cdot, \cdot)$ fails to be a symplectic form on $\bar{\Gamma}$, the space of solutions to the nonlinear field equations, because it is *degenerate*; equivalently, for each $s_b \in \bar{\Gamma}$, the set Γ_C consisting of classical solutions to equations (4.1) is unsuitable to serve as phase space of linear perturbation theory because it is "too large."

However, with the help of a complete set of basic gauge-invariant variables, we can try to prove that $\theta(s_b|s', \bar{s}')$ depends only on $\omega := \varphi([s'])$, i.e., that there exists a bilinear functional $\Theta(s_b|\omega, \bar{\omega})$ of $\omega \in \Omega_C$ and $\bar{\omega} \in \Omega_C$ related to $\theta(s_b|s', \bar{s}')$ by

$$\Theta(s_b|\omega, \bar{\omega}) = \theta(s_b|s', \bar{s}') \tag{6.2}$$

If this reduction process gives rise to a *symplectic structure* $\Theta(s_b|\cdot, \cdot)$: $\Omega_C \times \Omega_C \to \mathrm{R}$ via $(\omega, \bar{\omega}) \mapsto \Theta(s_b|\omega, \bar{\omega})$, it will be possible to find a quantum theory in which the functions $\Theta(s_b|\omega, \cdot)$ on the "classical phase space" $\Omega_C := \varphi(\Gamma_C/\Gamma_L)$ are represented (irreducibly) by operators $\hat{\Theta}(s_b|\omega, \cdot)$ satisfying the following commutation relations (Wald, 1994, p. 37):

$$[\hat{\Theta}(s_b|\omega, \cdot), \hat{\Theta}(s_b|\bar{\omega}, \cdot)] = -i\Theta(s_b|\omega, \bar{\omega})\hat{I} \tag{6.3}$$

where $\hat{I}$ denotes the identity operator and where we choose units where $\hbar = 1$.

The above discussion has laid out the basic mathematical framework of the geometric formulation of quantum field theory in a semiclassical approach.

However, as noted already by Wald (1994), the issue of how to make physical predictions from the theory for the outcomes of measurements remains to be addressed. We hope to study these and similar problems in the future.

APPENDIX. SOME CONVENTIONS REGARDING THE DEFINITION OF $\tilde{f}_{A,p} \odot \nabla^p(s^A)'$

We have already remarked that, given the result $\tilde{f}_{A,p} \odot \nabla^p(s^A)'$ of the contraction of $\tilde{f}_{A,p}$ with $\nabla^p(s^A)'$, generally some convention as to which of the $1 + 2(r_A + R_A + p)$ indices in a coordinate representation of $\tilde{f}_{A,p} \otimes \nabla^p(s^A)'$ are to be contracted must be followed when doing the contraction. For equation (2.8), the meaning of the symbol $\odot$ can be explained in several steps.

(A) If $\{e_a\}$ is a frame of the tensor bundle of type (1, 0) over X (1 is the contravariant index and 0 is the covariant index; $\dim X = N$), then there exists a unique frame $\{e^a\}$ of the tensor bundle of type (0, 1) over X such that $e^a \odot e_b = \gamma^a{}_b$ ($\gamma^a{}_b$ is the Kronecker delta and a and b are integers ranging from 1 to N). The frame $\{e^a\}$ is called the dual of $\{e_a\}$.

(B) Let us set

$$E^p_\beta := e^{a_1} \otimes \cdots \otimes e^{a_{R_A+p}} \otimes e_{b_1} \otimes \cdots \otimes e_{b_{r_A}} \tag{A.1a}$$

and define E^α_p by

$$E^\alpha_p := e_{c_1} \otimes \cdots \otimes e_{c_{R_A+p}} \otimes e^{d_1} \otimes \cdots \otimes e^{d_{r_A}} \tag{A.1b}$$

With the same notation as in Banach and Piekarski (1997, Section 2.1), from the definitions (A.1a) and (A.1b) we easily see that $\{E^p_\beta\}$ is a frame of the tensor bundle S^A_p and $\{E^\alpha_p\}$ is a frame of the tensor bundle S^{A*}_p dual to S^A_p. Clearly, the value of E^α_p on E^p_β, denoted $E^\alpha_p \odot E^p_\beta$ or $\delta^\alpha{}_\beta$, is 1 if $\alpha := (c_1 \cdots c_{R_A+p} d_1 \cdots d_{r_A})$ equals $\beta := (a_1 \cdots a_{R_A+p} b_1 \cdots b_{r_A})$ and is 0 otherwise.

(C) Since the objects $f_{A,p}$ and $\nabla^p(s^A)'$ appearing in equations (2.1) and (2.4) are, respectively, the cross sections of S^{A*}_p and S^A_p, these objects can be decomposed as

$$f_{A,p} = \sum_\alpha (f_{A,p})_\alpha E^\alpha_p \tag{A.2a}$$

$$\nabla^p(s^A)' = \sum_\beta (\nabla^p(s^A)')^\beta E^p_\beta \tag{A.2b}$$

where $(f_{A,p})_\alpha$ are the components of $f_{A,p}$ with respect to $\{E^\alpha_p\}$ and $(\nabla^p(s^A)')^\beta$ are the components of $\nabla^p(s^A)'$ with respect to $\{E^p_\beta\}$.

(D) Substitution of the decomposition (A.2a) for $f_{A,p}$ and a similar decomposition for $f_{A,p-1}$ into equation (2.9) yields the following result for $\tilde{f}_{A,p}$:

$$\tilde{f}_{A,p} = \sum_{a,b} \sum_\lambda (\tilde{f}_{A,p})^b_{a\lambda} e^a \otimes e_b \otimes E^\lambda_{p-1} \tag{A.3}$$

where

$$(\tilde{f}_{A,p})^b_{a\lambda} := (1 - \delta_{0,p})\gamma^b{}_a(f_{A,p-1})_\lambda + (1 - \delta_{r+1,p})\nabla_a[(f_{A,p})^b_\lambda] \tag{A.4}$$

with $(f_{A,p})^b_\lambda$ denoting the components of $f_{A,p}$ with respect to $\{e_b \otimes E^\lambda_{p-1}\}$. As regards ∇_a, the meaning of this symbol is conventional and is explained, e.g., in Wald (1984, pp. 30–36).

(E) If we define the contraction of $e^a \otimes e_b \otimes E^\lambda_{p-1}$ with $e^c \otimes E^{p-1}_\sigma$ by

$$(e^a \otimes e_b \otimes E^\lambda_{p-1}) \odot (e^c \otimes E^{p-1}_\sigma) = \gamma^c{}_b\delta^\lambda{}_\sigma e^a \tag{A.5}$$

we immediately obtain from (A.3) and

$$\nabla^p(s^A)' = \sum_c \sum_\sigma (\nabla^p(s^A)')^\sigma_c e^c \otimes E^{p-1}_\sigma \tag{A.6}$$

that

$$\tilde{f}_{A,p} \odot \nabla^p(s^A)' = \sum_{a,b} \sum_\lambda (\tilde{f}_{A,p})^b_{a\lambda}(\nabla^p(s^A)')^\lambda_b e^a \tag{A.7}$$

This completes the construction of $\tilde{f}_{A,p} \odot \nabla^p(s^A)'$ in the case when $f_{A,p} \odot \nabla^p(s^A)'$ is a real-valued field on X.

The notion at which we arrive in this way geometrically appears to be intrinsic: for it can be verified that, although the components $(\tilde{f}_{A,p})^b_{a\lambda}$ and $(\nabla^p(s^A)')^\sigma_c$ of $\tilde{f}_{A,p}$ and $\nabla^p(s^A)'$ depend on the choice of $\{e_a\}$, nevertheless the expression on the right-hand side of equation (A.7), called the value of $\tilde{f}_{A,p}$ on $\nabla^p(s^A)'$ or the contraction of $\tilde{f}_{A,p}$ with $\nabla^p(s^A)'$, does not depend on the particular frame $\{e_a\}$ chosen.

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